

Chromatic Vertex Folkman Numbers [☆]

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Abstract

For graph G and integers $a_1 \geq \cdots \geq a_r \geq 2$, we write $G \rightarrow (a_1, \dots, a_r)^v$ if and only if for every r -coloring of the vertex set $V(G)$ there exists a monochromatic K_{a_i} in G for some color $i \in \{1, \dots, r\}$. The vertex Folkman number $F_v(a_1, \dots, a_r; s)$ is defined as the smallest integer n for which there exists a K_s -free graph G of order n such that $G \rightarrow (a_1, \dots, a_r)^v$. It is well known that if $G \rightarrow (a_1, \dots, a_r)^v$ then $\chi(G) \geq m$, where $m = 1 + \sum_{i=1}^r (a_i - 1)$. In this paper we study such Folkman graphs G with chromatic number $\chi(G) = m$, which leads to a new concept of chromatic Folkman numbers. We prove constructively some related existential results. We also conjecture that, in some cases, our construction is the best possible, in particular that for every s there exists a K_{s+1} -free graph G on $F_v(s, s; s+1)$ vertices with $\chi(G) = 2s-1$ such that $G \rightarrow (s, s)^v$.

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1. Preliminaries, notation and definitions

Throughout this paper, we consider only finite undirected loopless simple graphs. For graph $G = (V, E)$, denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its edges. A complete graph of order n is denoted by K_n , and a cycle of length n by C_n . The clique number of G is denoted by $cl(G)$, and the chromatic number by $\chi(G)$. An (s, t) -graph is a graph that contains neither an s -clique nor a t -independent set.

The set $\{1, \dots, n\}$ is denoted by $[n]$. Let r, s, a_1, \dots, a_r be integers such that $r \geq 2$ and $s > \max\{a_1, \dots, a_r\}$ and $\min\{a_1, \dots, a_r\} \geq 2$. We write $G \rightarrow (a_1, \dots, a_r)^v$ (resp. $G \rightarrow (a_1, \dots, a_r)^e$) if and only if for every r -coloring of $V(G)$ (resp. $E(G)$), there exists a monochromatic K_{a_i} in G for some color $i \in \{1, \dots, r\}$. The Ramsey number $R(a_1, \dots, a_r)$ is defined as the smallest integer n such that $K_n \rightarrow (a_1, \dots, a_r)^e$.

The sets of vertex- and edge Folkman graphs are defined as

$$\mathcal{F}_v(a_1, \dots, a_r; s) = \{G \mid G \rightarrow (a_1, \dots, a_r)^v \text{ and } cl(G) < s\}, \text{ and}$$

$$\mathcal{F}_e(a_1, \dots, a_r; s) = \{G \mid G \rightarrow (a_1, \dots, a_r)^e \text{ and } cl(G) < s\},$$

respectively, and the vertex- and edge Folkman numbers are defined as the smallest orders of graphs in these sets, namely

$$F_v(a_1, \dots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_v(a_1, \dots, a_r; s)\}, \text{ and}$$

$$F_e(a_1, \dots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_e(a_1, \dots, a_r; s)\}.$$

In 1970, Folkman [6] proved that for $s > \max\{a_1, \dots, a_r\}$ the sets $\mathcal{F}_v(a_1, \dots, a_r; s)$ and $\mathcal{F}_e(a_1, a_2; s)$ are nonempty. In 1976, this result was generalized to $\mathcal{F}_e(a_1, \dots, a_r; s)$ for arbitrary $r \geq 2$ by Nešetřil and Rödl [15]. If $a_1 = \dots = a_r = a$, then we will use a simpler notation for the corresponding set of vertex Folkman graphs, $\mathcal{F}(r, a, s) = \mathcal{F}_v(a_1, \dots, a_r; s)$, and numbers $F(r, a, s) = F_v(a_1, \dots, a_r; s)$. The following theorem states the best known upper bound on $F(r, a, s)$, which was obtained by Dudek and Rödl [3].

Theorem 1. (Dudek-Rödl, 2010) *For any positive integer r there exists a constant $C = C(r)$ such that for every $s \geq 2$ it holds that $F(r, s, s + 1) \leq Cs^2 \log^4 s$.*

Set $m = 1 + \sum_{i=1}^r (a_i - 1)$ and $M = R(a_1, \dots, a_r)$, i.e. M is the classical Ramsey number. It is well known that if $G \rightarrow (a_1, \dots, a_r)^v$ then $\chi(G) \geq m$ [13], and similarly, if $G \rightarrow (a_1, \dots, a_r)^e$, then $\chi(G) \geq M$ [11]. In this paper we study mainly vertex- (and to much lesser extent edge-) Folkman graphs G with the corresponding Folkman numbers, when the graphs G satisfy an additional constraint on their chromatic numbers that they are the smallest possible, m or M , respectively. This motivates the concept of the *minimum chromatic Folkman graphs*, $\mathcal{F}_{v/e}^\chi(a_1, \dots, a_r; s)$, and corresponding *minimum chromatic Folkman numbers*, $F_{v/e}^\chi(a_1, \dots, a_r; s)$, both for vertex-coloring and edge-coloring. Formally, they are defined as follows:

$$\mathcal{F}_v^\chi(a_1, \dots, a_r; s) = \{G \mid G \in \mathcal{F}_v(a_1, \dots, a_r; s), \text{ and } \chi(G) = m\},$$

$$\mathcal{F}_e^\chi(a_1, \dots, a_r; s) = \{G \mid G \in \mathcal{F}_e(a_1, \dots, a_r; s), \text{ and } \chi(G) = M\},$$

and

$$F_v^\chi(a_1, \dots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_v^\chi(a_1, \dots, a_r; s)\},$$

$$F_e^\chi(a_1, \dots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_e^\chi(a_1, \dots, a_r; s)\},$$

where m and M are defined as above.

Recall that if $G \rightarrow (s, s)^e$, then $\chi(G) \geq R(s, s)$. From the construction in [15] we can see that there exists a K_{s+1} -free graph G with chromatic number $R(s, s)$ such that $G \rightarrow (s, s)^e$. For $3 \leq k \leq l$, essentially the same reasoning as in [15] implies the existence of K_{s+1} -free graphs G with $\chi(G) = R(k, l)$ and $G \rightarrow (k, l)^e$. In a similar way, one can also easily show that $F_e^\chi(a_1, \dots, a_r; s)$ exists, thus answering in positive the existence question of minimum chromatic edge Folkman numbers.

In the sequel we will be coloring the vertices. The remainder of this paper focuses on minimum chromatic vertex Folkman graphs and numbers, and we will call them simply chromatic Folkman graphs and numbers, respectively. We will use further notational abbreviation for the diagonal cases, namely, if $a_1 = \dots = a_r = a$, then we set $\mathcal{F}^\chi(r, a, s) = \mathcal{F}_v^\chi(a_1, \dots, a_r; s)$ and $F^\chi(r, a, s) = F_v^\chi(a_1, \dots, a_r; s)$.

The main result of this paper is the existence of $F^\chi(r, s, s+1)$ obtained by an explicit construction. We also conjecture that, in some cases, our construction is the best possible one, in particular that for every s there exists a K_{s+1} -free graph G on $F_v(s, s; s+1)$ vertices with chromatic number $\chi(G)$ equal to $m = 2s - 1$ such that $G \rightarrow (s, s)^v$, or equivalently, that $F_v^\chi(s, s; s+1) = F_v(s, s; s+1)$. We wish to remark that a technique as in this paper leads to more general existence results for $F_v^\chi(a_1, \dots, a_r; s)$, and with some further enhancements also for generalized Folkman numbers, where one avoids arbitrary graphs G_i instead of K_{a_i} . However, we do not study such extensions in this work.

The rest of this paper is organized as follows. In Section 2, the existence of the chromatic Folkman number $F^\chi(r, s, s+1)$ is proved, the corresponding upper bound is discussed, and a conjecture is posed. In Section 3, we study some minimal Folkman graphs with the smallest minimum degree. Though not directly connected to chromatic Folkman numbers, we point to a possible connection between them.

2. Chromatic Folkman numbers

Before we prove the existence of $F^\chi(r, s, s+1)$, we want to observe that $F_v^\chi(a_1, \dots, a_r; s)$ and $F_v(a_1, \dots, a_r; s)$ can be different. It is known that there exists exactly one K_5 -free graph G of order 13 such that $G \rightarrow (3, 4)^v$ [16]. This graph G is the complement of the unique $(3, 5)$ -Ramsey critical graph, which is cyclic on the set \mathcal{Z}_{13} with arcs of length 1 and 5. One can easily check that $\chi(G) = 7$. Thus, since in this case $m = 6$, we have $F_v^\chi(3, 4; 5) > 13 = F_v(3, 4; 5)$. Is this the only counterexample, is the number of such counterexamples finite? What about more colors? We state these questions as the following problems.

Problem 1. *Is it true that $F_v^\chi(a_1, a_2; s) = F_v(a_1, a_2; s)$ for all $s > a_1, a_2 \geq 2$, except the case $(a_1, a_2, s) \neq (3, 4, 5)$?*

Problem 2. *Is it true that $F^\chi(r, a, s) = F(r, a, s)$ for all $r, a \geq 2$ and $s > a$?*

Note that the exception to Problem 1 above is not a counterexample to Problem 2, because it is not on the diagonal. One can ask similar questions related to minimum chromatic edge Folkman numbers, but these seem much more difficult to answer.

While several general cases of vertex Folkman numbers have been studied, the problem of finding their exact values remains elusive, especially when s is smaller than $m - 1$. For instance, even a small case like $F_v(4, 4; 5)$ seems to be difficult, for which only the bounds $17 \leq F_v(4, 4; 5) \leq 23$ are known [16]. The computational approach is often too expensive. Just testing a single instance of arrowing for an upper bound witness graph is not easy in most cases, and improving lower bounds is much harder since it may involve a very large number of arrowing instances. Both upper and lower bounds for edge Folkman numbers tend to be computationally even harder.

2.1. Main theorem

The upper bound on $F^x(r, s, s + 1)$, which can be obtained by the construction of Theorem 3 below, is rather large. This and other bounds we could derive for $F^x(r, a, s)$ also seem less tight than those known for $F(r, a, s)$, except the special case captured by Lemma 2. We follow this lemma by a theorem describing quite special but more general and more difficult case of $F^x(r, s, s + 1)$.

Lemma 2. *For all $r \geq 2$, $F^x(r, 2, 3)$ exists and it is equal to $F(r, 2, 3)$.*

Proof. The sets $\mathcal{F}^x(r, 2, 3)$ consist of triangle-free graphs with chromatic number equal to $r + 1$, which in the case of arrowing K_2 clearly coincides with the smallest order graphs in $\mathcal{F}_v(r, 2, 3)$. The latter sets are known to be nonempty for all $r \geq 2$. Thus we also have $F^x(r, 2, 3) = F(r, 2, 3)$. \square

In the basic case of two colors, one can easily see that $F^x(2, 2, 3) = 5$, because $C_5 \in \mathcal{F}_v(2, 2; 3)$ and $\chi(C_5) = m = 3$. Clearly, we also have $F_v(2, 2; 3) = 5$. We know that $F_v(2, 2, 2; 3) = 11$, or the smallest 4-chromatic triangle-free graph has 11 vertices, witnessed by the Grötzsch graph. Also, it is known that $F^x(4, 2, 3) = F_v(2, 2, 2, 2; 3) = 22$, or the smallest 5-chromatic triangle-free graph has 22 vertices [8].

Theorem 3. *For given integers $r \geq 2$ and $s \geq 3$, let $b_i = i(s - 1) + 1$ for $i \in [r - 1]$, and $B = \prod_{i=1}^{r-1} b_i$. Then $F^x(r, s, s + 1)$ exists and*

$$F^x(r, s, s + 1) \leq 1 + s + \sum_{i=2}^{r-1} F^x(i, s, s + 1) + B \cdot F^x(r, s - 1, s). \quad (1)$$

In particular, for all $s \geq 3$, the chromatic Folkman number $F^x(2, s, s + 1)$ exists and we have $F^x(2, s, s + 1) \leq 1 + s + sF^x(2, s - 1, s)$.

Proof. We will construct a graph $G(r, s) \in \mathcal{F}^\chi(r, s, s+1)$ given any graphs in each of $\mathcal{F}^\chi(i, s, s+1)$ for $2 \leq i \leq r-1$ and any graph in $\mathcal{F}^\chi(r, s-1, s)$. The vertices of the graph $G(r, s)$ will be formed by vertices of given graphs corresponding to the terms of the right-hand-side of (1). The proof is using simultaneous induction on r and s , and it has two main parts: construction of $G(r, s)$, and the proof that $G(r, s)$ has required properties.

Note that the second part of the theorem is just an instantiation of the first part for two colors, $r = 2$, in which case the main summation of (1) is empty. Thus the basis of our induction is formed by the sets $\mathcal{F}^\chi(i, 2, 3)$, which are nonempty by Lemma 2, and the corresponding vertex Folkman numbers $F(i, 2, 3)$.

Construction of the graph $G(r, s)$ by induction.

Let G_0 be the graph of order 1, $G_1 = K_s$, and set $V_0 = V(G_0)$, $V_1 = V(G_1)$. We may assume that the graphs $G_i = (V_i, E_i)$ such that $|V_i| = F^\chi(i, s, s+1)$ and $G_i \in \mathcal{F}^\chi(i, s, s+1)$ have been already constructed, for $2 \leq i < r$. Therefore, we know that $\chi(G_i) = b_i = i(s-1) + 1$ for $i \in \{0, \dots, r-1\}$, and hence we can partition each of the sets of vertices V_i into $\chi(G_i)$ nonempty independent sets $V_i(j)$ in G_i , so that

$$V_i = \bigcup_{j=1}^{\chi(G_i)} V_i(j).$$

Let H be any graph in the set $\mathcal{F}^\chi(r, s-1, s)$ with $|V(H)| = F^\chi(r, s-1, s)$ vertices. For the part of $G(r, s)$ corresponding to the last term of (1), we take $B = \prod_{i=1}^{r-1} b_i$ isomorphic copies $H(j_0, \dots, j_{r-1})$ of H indexed by r -tuples (j_0, \dots, j_{r-1}) , where $1 \leq j_k \leq \chi(G_k)$ for $k \in \{0, \dots, r-1\}$. Note that the order of $G(r, s)$ is equal to the right-hand-side of (1) because the sizes of parts described above match exactly its terms,

$$V = V(G(r, s)) = V_0 \cup V_1 \cup \bigcup_{i=2}^{r-1} V_i \cup \bigcup_{(j_0, \dots, j_{r-1})} V(H(j_0, \dots, j_{r-1})).$$

Finally, we complete the construction of $G(r, s)$ by adding all possible edges with one end in any of the sets of vertices $V(H(j_0, \dots, j_{r-1}))$ and the other end in the sets $V_i(j_i)$, for $i \in \{0, \dots, r-1\}$ and $1 \leq j_i \leq b_i$.

Proof that $G(r, s) \in \mathcal{F}^\chi(r, s, s+1)$.

We need to show that: (i) $cl(G(r, s)) < s+1$, (ii) in every r -coloring of the vertices V we have a monochromatic K_s , and (iii) $\chi(G(r, s)) = m = r(s-1) + 1$.

(i) The graph $G(r, s)$ contains $G_1 = K_s$, so $cl(G(r, s)) \geq s$. Suppose that some set $S \subset V$ of order $s+1$ induces K_{s+1} , and let $k_i = |S \cap V_i|$ for $0 \leq i < r$. From the construction we see that $k_i \leq s$ and there exists exactly one t for which $k_t > 0$. Similarly, there exists exactly one r -tuple (j_0, \dots, j_{r-1}) such that $h = |S \cap V(H(j_0, \dots, j_{r-1}))| > 0$. Note that since $h \leq s-1$ and $k_t + h = s+1$, then $k_t \geq 2$. However, each vertex of $H(j_0, \dots, j_{r-1})$ can be adjacent only to independent sets $V_i(j) \subset V_i$, hence we have a contradiction for $i = t$. Thus $cl(G(r, s)) = s$.

(ii) Consider any r -coloring C of V . Let c_0 be the color of V_0 , and c_1 be a different color of one of the vertices in V_1 . From the inductive assumption that $\chi(G_i) = b_i$ and the construction, we can find a vertex $v_i \in V_i$ in a new color $c_i = C(v_i)$, for each $2 \leq i < r$. Let i' be such that $v_i \in V_i(i')$.

We know that each graph $H(j_0, \dots, j_{r-1})$ is isomorphic to $H \in \mathcal{F}^\chi(r, s-1, s)$ and its order is $F^\chi(r, s-1, s)$, and therefore it has a monochromatic $(s-1)$ -clique S in color c , where $c_i = c$ for some i . Recall that for each i , all $\chi(G_i)$ independent sets $V_i(j)$ are nonempty. Consequently, S can be extended to a clique of order s in color c_i by any vertex in $V_i(i')$. Hence, $G(r, s) \in \mathcal{F}(r, s, s+1)$.

(iii) Part (ii) implies that $\chi(G(r, s)) \geq r(s-1) + 1$, hence we only need to prove that $\chi(G(r, s)) \leq r(s-1) + 1$. We will do that by showing how to color appropriately V with $r(s-1) + 1$ colors.

First, for each $0 \leq i < r$, we color the vertices in $V_i(j)$ with color j , for $1 \leq j \leq \chi(G_i)$. This step, by the inductive assumption, uses exactly $(r-1)(s-1) + 1$ colors. Similarly, note that each of $H(j_0, \dots, j_{r-1})$ could be properly colored by itself with $r(s-2) + 1$ colors, r less than the total number of colors allowed. Observe however that r new colors are sufficient for proper recoloring of $H(j_0, \dots, j_{r-1})$ as a part of G , namely one for each j_t . This completes the proof of the theorem. \square

It is interesting to see that if the vertex of V_0 is removed from $G(r, s)$, then in the part (iii) of the proof of Theorem 2 $\chi(G(r, s)) = m$ drops to $m-1$, and thus also the arrowing of part (ii) does not hold.

2.2. Upper bounds on F_v^χ

Theorem 4. *For any integers a, b and s such that $2 \leq a, b \leq s$, we have*

$$F_v^\chi(a, b; s+1) \leq \frac{a+b-1}{2s-1} F_v^\chi(s, s; s+1).$$

Proof. Suppose that $G \in \mathcal{F}_v^\chi(s, s; s+1)$, and the order of G is $F_v^\chi(s, s; s+1)$. Therefore $\chi(G) = 2s-1$. Write the set of vertices of G as a partition

$$V(G) = \bigcup_{j=1}^{2s-1} I_j,$$

where I_j 's are independent sets for $j \in \{1, \dots, 2s-1\}$, and $|I_{j_1}| \leq |I_{j_2}|$ for $j_1 < j_2$ and $j_1, j_2 \in \{1, \dots, 2s-1\}$. Let G_i be the subgraph of G induced by $\bigcup_{j=1}^i I_j$. Note that this implies $\chi(G_i) = i$ for each i , since otherwise $\chi(G) < 2s-1$.

We claim that $G_{a+b-1} \rightarrow (a, b)^v$. For a contradiction suppose that we have a red-blue coloring of $V(G_{a+b-1})$ without any red K_a and without any blue K_b . We can extend this coloring to a full red-blue coloring of $V(G)$ by coloring red all the vertices in I_j 's for $a+b \leq j \leq s+b-1$, and coloring blue all the vertices in I_j 's for $s+b \leq j \leq 2s-1$. This coloring does not contain any monochromatic K_s , which contradicts the assumption that $G \in \mathcal{F}_v^\chi(s, s; s+1)$. Considering the non-decreasing orders of the sets I_j , we have that

$$|V(G_i)| \leq \frac{a+b-1}{2s-1} F_v^\chi(s, s; s+1),$$

and thus the theorem follows. \square

Bollobás and Thomason [1] studied the set-coloring of graphs, where an r -set-coloring of a graph G is defined as an assignment of r distinct colors to each vertex of G so that the sets of colors assigned to adjacent vertices are disjoint. The set-coloring variants of the vertex- and edge Folkman numbers were previously introduced and studied by the first two authors of this paper jointly with Wenfei Zhao and Zehui Shao [17].

Let us denote the minimum number of colors required to r -set-color any given graph G by $\chi^{(r)}(G)$. In 1979, Bollobás and Thomason proved that $\min\{\chi^{(r)}(G) \mid \chi(G) = t\} = t + 2r - 2$ [1]. We need a simple lemma using this result as follows.

Lemma 5. $\chi^{(2s-1)}(C_{4s-1}) = 4s - 1$.

Proof. Using the above result by Bollobás and Thomason on $\chi^{(r)}(G)$, since $\chi(C_{4s-1}) = 3$, we clearly have $\chi^{(2s-1)}(C_{4s-1}) \geq 4s - 1$. On the other hand, it is easy to give a proper set-coloring witnessing $\chi^{(2s-1)}(C_{4s-1}) \leq 4s - 1$. We take both the vertices of the cycle and colors to be in the set \mathcal{Z}_{4s-1} , and assume that the edges of the cycle are $\{i, i+1\}$ for $i \in \mathcal{Z}_{4s-1}$, all modulo $4s - 1$. We assign the colors $\{i(2s-1) + j \mid 0 \leq j \leq 2s-2\}$ to the vertex i of C_{4s-1} , for each $i \in \mathcal{Z}_{4s-1}$. One can easily see that the sets of colors assigned to adjacent vertices are disjoint. \square

The composition of simple graphs G and H is denoted by $G[H]$, and it is defined as the graph with vertex set $V(G) \times V(H)$, in which vertex (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. We will need another simple lemma involving $G[H]$ and $\chi^{(r)}(G)$, namely:

Lemma 6. *If G and H are graphs and $\chi(H) = r$, then $\chi^{(r)}(G) = \chi(G[H])$.*

Proof. Klavžar proved that if $\chi(H) = r$, then $\chi(G[H]) = \chi(G[K_r])$ [9]. We can also easily see that $\chi^{(r)}(G) = \chi(G[K_r])$. The lemma follows. \square

The bound $F_v(2s, 2s; 2s+1) \leq 5F_v(s, s; s+1)$ was obtained constructively by Konev who used the composition of graphs $G[H]$ [10]. We will use a similar approach to obtain an upper bound on the chromatic Folkman numbers of the form $F_v^\chi(2s, 2s; 2s+1)$.

Theorem 7. *For any integer $s \geq 2$, we have*

$$F_v^\chi(2s, 2s; 2s+1) \leq (4s-1)F_v^\chi(s, s; s+1).$$

Proof. Let H be any graph in $\mathcal{F}_v^\chi(s, s; s+1)$ of order $F_v^\chi(s, s; s+1)$, and thus $\chi(H) = 2s - 1$. Observe that $C_{4s-1}[H] \rightarrow (2s, 2s)^v$. By Lemmas 5 and 4, we have $\chi(C_{4s-1}[H]) = \chi^{(2s-1)}(C_{4s-1}) = 4s - 1$. This in turn implies that $C_{4s-1}[H] \in \mathcal{F}_v^\chi(2s, 2s; 2s+1)$. Finally, since the order of $C_{4s-1}[H]$ is equal to $(4s-1)F_v^\chi(s, s; s+1)$ and clearly $F_v^\chi(2s, 2s; 2s+1) \leq |V(C_{4s-1}[H])|$, this completes the proof. \square

2.3. A conjecture

The classical Turán graph $T_{n,r}$ is a complete multipartite graph on n vertices whose r partite sets have sizes as equal as possible.

Conjecture 1. *For any integer $s \geq 3$, let $n = F_v(s, s; s + 1)$. Then there exists an n -vertex K_{r+1} -free subgraph G of the Turán graph $T_{n, 2s-1}$, such that $G \rightarrow (s, s)^v$.*

Any subgraph of $T_{n, 2s-1}$ has the chromatic number upper bounded by $2s - 1$. Therefore, we can easily see that Conjecture 1 implies the equality $F_v^x(s, s; s + 1) = F_v(s, s; s + 1)$. If proved true, it would make the search for the upper bound witnesses much easier, including an approach using computer constructions.

Nenov studied several problems related to $F_v(r, 2, s + 1)$, for instance in [14]. Observe that using essentially only the definitions, we can see that $F_v(2s - 2, 2, s + 1) \leq F_v^x(s, s; s + 1)$, though we suspect that much better upper bound on $F_v(2s - 2, 2, s + 1)$ is true. Independently, $F_v(2s - 2, 2, s + 1)$ may be much smaller than $F_v(s, s; s + 1)$. In another direction, one could use C_5 similarly as we used C_{4k-1} in Theorem 6, to study the cases of $F_v^x(2s, 2s, 2s; 2s + 1)$, and more general diagonal and non-diagonal cases. All of these problems seem interesting but difficult.

3. Minimum degree of graphs in $\mathcal{F}_v(s, s; s + 1)$

In this section we prove a theorem and then pose a problem concerning a lower bound on the minimum degree in some minimal Folkman graphs. We consider only the case of $F_v(s, s; s + 1)$, but one can also easily see that some other parameter situations can be approached similarly as in the proof of the following theorem.

Theorem 8. *For all integers $s \geq 3$, we have:*

- (a) *For every graph G , if $G \rightarrow (s, s)^v$ and $G - u \not\rightarrow (s, s)^v$ for every vertex $u \in V(G)$, then the minimum degree $\delta(G)$ satisfies $\delta \geq 2s - 2$, and*
- (b) *There exists a K_{s+1} -free graph G with minimum degree $\delta(G) = 2s - 2$, such that $G \rightarrow (s, s)^v$ and $G - u \not\rightarrow (s, s)^v$ for every vertex $u \in V(G)$.*

Proof. (a) For contradiction, let $u \in V(G)$ be any vertex of degree at most $2s - 3$. Assuming that $G - u \not\rightarrow (s, s)^v$, consider any K_s -free bipartition $V_1 \cup V_2$ of the remaining vertices, so that $V(G) = V_1 \cup V_2 \cup \{u\}$ and $|V_1| \leq s - 2$. Color vertices in $V_1 \cup \{u\}$ red and those in V_2 blue, and note that this coloring has no monochromatic K_s . This contradicts $G \rightarrow (s, s)^v$, and thus (a) follows.

(b) Let H be any K_{s+1} -free graph such that $H \rightarrow (s, s)^v$. Assume further that H is both vertex- and edge-minimal with respect to arrowing, that is to say, $H - u \not\rightarrow (s, s)^v$ for all vertices $u \in V(H)$ and $H - e \not\rightarrow (s, s)^v$ for all edges $e \in E(H)$.

For any vertex $u \in V(H)$, let $N_H(u)$ denote the set of all neighbors of u in $V(H)$. Fix some vertex $u_0 \in V(H)$, and let A be the set of all $(s-2)$ -element subsets of $N_H(u_0)$ that contain two vertex-disjoint K_{s-1} . Suppose that A consists of m sets, i.e. $A = \{V_i \mid 1 \leq i \leq m\}$. First we extend graph H to H' by adding new vertices $\{u_i \mid 1 \leq i \leq m\}$ and edges $\{u_i v \mid v \in V_i\}$ for all $1 \leq i \leq m$. Next, we delete vertex u_0 from H' . Observe that for all $1 \leq i \leq m$ the degree of vertex u_i in H' is equal to $2s-2$.

We claim that $H' \rightarrow (s, s)^v$. Suppose the contrary, namely that there exists a red-blue coloring C' of $V(H')$ without monochromatic K_s . Thus, the restriction of coloring C' to coloring the vertices of $H - u_0$, say C , is a witness of $H - u_0 \not\rightarrow (s, s)^v$. Since $H \rightarrow (s, s)^v$, C must contain two vertex-disjoint monochromatic K_{s-1} , furthermore in different colors. This however contradicts the properties of C' following from the construction of H' , and hence $H' \rightarrow (s, s)^v$. Note that we also must have $m \geq 1$.

We will define the final graph G as in (b) to be an induced subgraph of H' on the vertex set of the form $V(H') \setminus B$, where $B \subset \{u_i \mid 1 \leq i \leq m\}$. We choose B so that its vertex indices form a maximal subset of $[m]$ still giving $G \rightarrow (s, s)^v$. The properties of H' stated above guarantee that such B must be a proper subset of $[m]$. This can be seen, since if we delete $\{u_i \mid 1 \leq i \leq m\}$, we obtain a graph isomorphic to $H - u_0$, which does not arrow $(s, s)^v$. This way, we can obtain a minimal graph G which arrows $(s, s)^v$ with at least one vertex u_i of degree $2s-2$. Together with part (a), this completes the proof of (b). \square

Finally, we pose the following question.

Problem 3. *For which integers $n, s \geq 3$ does there exist a $(2s-2)$ -regular K_{s+1} -free graph G on n vertices such that $G \rightarrow (s, s)^v$?*

It seems that even the case of $s = 3$ is not obvious. If the answer to this problem for each s is yes for at least some n , then the chromatic number of such a graph must be equal to $2s-1$. Thus, it could give another proof of the existence of $F_v^x(s, s; s+1)$. Of course, the order n of such a graph G may be much larger than $F_v^x(s, s; s+1)$. Possibly, however, such a graph G does not exist, at least for some s .

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